UNIT 12 : THE RIEMANN-STIELTJES INTEGRAL

UNIT STRUCTURE

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12.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- learn about a generalization of the Riemann integral
- know about lower and upper Riemann-Stieltjes integrals
- know about some classes of Riemann-Stieltjest integrable functions
- find the difference between Riemann integral and Riemann-Stieltjes integral.

12.2 INTRODUCTION

In this previous unit, we discussed about the theory of Riemann integration. In this unit, we generalize the theory and study the extensions due to Stieltjes, known as Riemann-Stieltjes integration. Throughout the unit, the function f we consider, will be real-valued bounded function on the domain of definition I : = [a, b]. Unlike the previous unit, here we introduce another monotonic increasing function α on [a, b].

12.3 GENERALIZATION OF RIEMANN INTEGRAL

As stated in the introduction, let f be a bounded function on [a, b] and α be a monotonic increasing function on [a, b].

We begin by defining

$\Delta\alpha_i = \alpha(\chi_i) - \alpha(\chi_{i-1}),\ i = 1, 2, ..., n$

corresponding to any partition $P := \{a = \chi_0, \chi_{1,.....}\chi_{n-1}, b = \chi_n\}$ of [a, b].

Since α is a monotonic increasing function, hence $\Delta\alpha_i \geq 0$

Let M_i and m_i be the supremum and infimum of f in (χ_{i-1}, χ_i)

We define Upper and Lower sum of f corresponding to partition P as follows :

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$$

and $$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

If M and m are the supremum and infimum of f in [a, b], then

$$m \leq m_i \leq M_i \leq M$$

which implies

$$m\,\Delta\alpha_i \leq \alpha_i \leq M_i\,\Delta\alpha_i \leq M\,\Delta\alpha_i$$

Putting i = 1, 2, ..., n and adding all the inequalities, we get

$$m\{a(b) - \alpha(a)\} \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq M\{a(b) - \alpha(a)\}$$

For each partition P, there exist L (P, f, α). We denote U to be the set of all upper sums of f for all partitions of [a, b] and L to be the set of all lower sums of f for all the partitions of [a, b]. Since both U (P, f, α) and L (P, f, α) are bounded as seen from (1), so the set U and L have supremum and infimum. The infimum of the set of upper sums is called the Upper integral of f with respect to α and the supremum of the set of lower sums is called the Lower integral of f with respect to α i.e.

$$\int_a^{\bar{b}} f\,d\alpha = \inf U$$

and $\underline{\int_a^b} f\,d\alpha = \sup L$

These two integrals may or may not be equal. Definition : The function f is said to be Integrable with respect to α in Riemann sense if

$$\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$$

and the common value of the integral is called the Riemann-Stieltjes integral of f with respect to α, over [a, b] denoted as $\overline{\int_a^b} f\,d\alpha$

Note : The upper and lower integrals are defined for every bounded function but they may not necessarily be equal for every bounded function. Functions whose upper and lower integrals are not equal are said to be non-integrable functions.

Note : If f is integrable with respect to α, then it can be expressed as $f \in R_\alpha[a, b]$ or simply R(α).

Note : The Riemann-Stieltjes integral reduces to Riemann integral when $\alpha(x) = x$.

Remark 1 : From (1) and definition of integral, we have

$$m\{a(b)-\alpha(a))\} \le L(P,f,\alpha) \le \underline{\int_a^b} f d\alpha \; \overline{\int_a^b} f d\alpha \le U(P,f,\alpha) \le M\{\alpha(b)-\alpha(a)\}$$

Refinement of Partitions

Like in the previous chapter, we denote P* to be the refinement of partition P. The following theorem is stated as follows :

Theorem : If P* is a refinement of P, then

(i) $L(P^*, f, \alpha) \ge L(P, f, \alpha)$

(ii) $U(P^*, f, \alpha) \le U(P, f, \alpha)$

Proof : (ii) Let p : = ($a = x_0, x_1, ...x_{n-1}, x_n = b$) be a partition of the given interval.

Let us suppose P* contains just one point more than P and let this extra point be ξ where $x_{i-1} < \xi < x_i$ for some i.

Since f is bounded over the interval [a , b], so it is bounded on

every sub-interval $[x_{i-1},\ x_i]\ \forall i = 1, 2, ..., n$

Let W1, W2, and Mi be the supremum of f in the intervals $[x_{i-1}, \xi]$, $[\xi, x_i]$, $[x_{i-1}, x_i]$ respectively.

Clearly $M_i \geq W_1$ and $M_i \geq W_2$.

Thus,

$$U(P^*, f, \alpha) = W_1(\alpha(\xi) - \alpha(x_{i-1})) + W_2(\alpha(x_i) - \alpha(\xi)) - M_i(\alpha(x_i) - \alpha(x_{i-1}))$$

$$= (W_1 - M_i)(\alpha(\xi) - \alpha(x_{i-1})) + (W_2 - M_i)(\alpha(x_i) - \alpha(\xi)) \leq 0$$

Thus, $U(P^*, f, \alpha) \leq U(P, f, \alpha)$

If P* contains m points more than P, same steps can be repeated m times to arrive at

$$U(P^*, f, \alpha) \leq U(P, f, \alpha)$$

Theorem : A function f is integrable with respect to α on [a, b] if an only if for every $\in > 0$ there exists a partition P of [a, b] such that

$U(P_2, f, \alpha) - L(P_1, f, \alpha) < \in$

Proof : Necessary Part : Let $f \in R(\alpha)$ over [a, b]

$$\int_a^b f d\alpha = \int_a^{\bar{b}} f d\alpha = \int_{\underline{a}}^b f d\alpha$$

Let $\in > 0$ be any number. Since the upper and the lower integrals are the infimum and the supremum, respectively, of the upper and the lower sums, therefore there exists partitions P_1 and P_2 such that

$$U(P_1, f, \alpha) < \int_a^{\bar{b}} f d\alpha + \frac{1}{2} \in = \int_a^b f d\alpha + \frac{1}{2} \in$$

$$L(P_2, f, \alpha) > \int_{\underline{a}}^b f dx\alpha - \frac{1}{2} \in = \int_a^b f d\alpha - \frac{1}{2} \in$$

Let $P = P_1 \cup P_2$ be the common refinement of P_1 and P_2.

$U(P, f, \alpha) - L(P, f, \alpha) < \in$

Sufficient part : Let $\in > 0$. For any partition P such that

$U(P, f, \alpha) - L(P, f, \alpha) < \in$

For any partition P, we know that

$$L(P, f, \alpha) \leq \int_{\underline{a}}^b f d\alpha \leq \int_a^{\bar{b}} f d\alpha \leq U(P, f, \alpha)$$

$$\Rightarrow \int_a^{\bar{b}} f d\alpha - \int_{\underline{a}}^b f d\alpha \leq U(P, f, \alpha) - L(P, f, \alpha) < \in$$

But $\in$ is an arbitrary positive number which can be as small as

possible but greater than 0, so

$$\int_a^{\bar{b}} f d\alpha - \int_{\underline{a}}^b f d\alpha$$

$$\Rightarrow \int_a^{\bar{b}} f d\alpha = \int_{\underline{a}}^b f d\alpha$$

Thus Let $f \in R(\alpha)$ over [a, b]



CHECK YOUR PROGRESS 1

Q 1: If $P \subseteq P^*$, then prove that

$$U(P^*, f, \alpha) - L(P^*, f, \alpha) \leq U(P, f, \alpha) - L(P, f, \alpha)$$

Q 2: For any two partitions P_1 and P_2.

$$L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$$

12.4 RIEMANN-STIELTJES INTEGRAL AS A LIMIT OF SUM

As an analog to the Riemann sum, in this section, we introduce the sum S (P, f, α) which will lead to a sufficient condition for the existence of a Riemann-Stieltjes integral.

Definition : We define the sum S (P, f, α) as

$$S(P, f, \alpha) = \sum_{i=1}^{n} f(t_i) \Delta \alpha_i$$

corresponding to a partition P of [a, b] and $t_i \in [x_{i-1} - x_i]$

Definition : We say that S (P, f, α) coverges to A as $\mu(P) \to 0$ i.e.

$$\lim_{\mu(P) \to 0} S(P, f, \alpha) = A$$

if for every $\in > 0$, there exists $\delta > 0$ such that for every partition P of [a, b] with $\mu(P) \to \delta$ and every choice to t_i in $[x_{i-1}, x_i]$.

Theorem : If exists, then $\lim_{\mu(P) \to 0} S(P, f, \alpha)$ exist, then

$$f \in R(\alpha) \text{ and } \lim_{\mu(P) \to 0} S(P, f, \alpha) = \int_a^b f d\alpha$$

Proof : Let us suppose $\lim_{\mu(P) \to 0} S(P, f, \alpha)$ exists and $\lim_{\mu(P) \to 0} S(P, f, \alpha)$ = A

Thus for $\in > 0$, there exists $\delta > 0$ such that for every partition P of [a, b] with $\mu(P) \to \delta$ and every choice of t_i and $[x_{i-1}, x_i]$, we have

$$\left|S(P,f,\alpha) - A\right| < (1/2)\in$$

or $$A - \frac{1}{2}\in < S(P,f,\alpha) < A + \frac{1}{2}\in$$

Let P be one such partition. If we let the points t_i range over the intervals $[x_{i-1}, x_i]$ and take the infimum and supremum of the sum S (P, f, α) then (6) gives,

$$A - \frac{1}{2}\in < L(P,f,\alpha) < U(P,f,\alpha) < A + \frac{1}{2}\in$$

$$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) < \in$$

$$\Rightarrow f \in R(\alpha) \text{ over [a, b]}$$

Again, since S (P, f, α) and $\int_a^b fd\alpha$ lie between $U(P,f,\alpha)$ and $L(P,f,\alpha)$, so we have

$$\left|S(P,f,\alpha) - \int_a^b fd\alpha\right| \le U(P,f,\alpha) - L(P,f,\alpha) < \in$$

$$\Rightarrow \lim_{\mu(P) \to 0} S(P,f,\alpha) = \int_a^b fd\alpha$$

12.5 SOME CLASSES OF RIEMANN-STIELTJES INTEGRABLE FUNCTIONS

Theorem : If f is continuous on [a, b], then $f \in R(\alpha)$ over [a, b]. Also, for every $\in > 0$ there corresponds $\delta > 0$ such that

$$\left|S(P,f,\alpha) - \int_a^b fd\alpha\right| < \in$$

for every partition P = {a = x_0, ..., x_n = b}of [a, b] with $\mu(P) \to \delta$ and for every choice of $t_i \in [x_{i-1} - x_i]$ i.e.

$$\lim_{\mu(P) \to 0} S(P,f,\alpha) = \int_a^b fd\alpha$$

Proof : If is assumed that all functions are bounded and α is monotonically increasing. Let $\in > 0$ be given. Let us choose $\eta > 0$ such that

$$\eta\{a(b) - a(\alpha)\} < \in$$

Since continuity of f on [a, b] implies uniform continuity on [a, b], therefore for $\eta > 0$, there exists $\delta > 0$ such that

$|f(t_1) - f(t_2)| < \eta$ if $|t_1 - t_2| < \delta, t_1, t_2 \in [a, b]$

Let P be a partition of [a, b] with $\mu(P) \to \delta$. Then from (7), we get

$M_i - m_i \le \eta,\ i = 1, 2, ...n$

Therefore,

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_i (M_i - m_i)\,\Delta x_i$$
$$\le \eta \sum_i \Delta x_i$$
$$= \eta[a(b) - \alpha(a)] < \in$$

Thus $f \in R(\alpha)$ over [a, b].

Again if $f \in R(\alpha)$, then for $\in > 0$, there exists $\delta > 0$ such that for all partitions P with $\mu(P) \to \delta$ and for all positions with $t_i \in [x_{i-1} - x_i]$, therefore we have

$$\left| S(P, f, \alpha) - \int_a^b f d\alpha \right| < U(P, f, \alpha) - L(P, f, \alpha) < \in$$

Thus we have

$$\lim_{\mu(P) \to 0} S(P, f, \alpha) = \lim_{\mu(P) \to 0} \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i = \int_a^b f d\alpha$$

Note : Continuity is a sufficient condition but not necessary condition for integrability of a function.



CHECK YOUR PROGRESS 2

Q 1: If f is monotonic on [a, b] and α is monotonic as well as continuous on [a, b] then $f \in R(\alpha)$.

12.6 A RELATION BETWEEN R-INTEGRAL AND RS-INTEGRAL

The following theorem provides a relation between the Riemann integral and Riemann-Stieltjes integral

Theorem : If $f \in R(\alpha)$ and α is monotonically increasing on [a, b] such that $a' \in R[a, b]$ then $f \in R(\alpha)$ and

$$\int_a^b f d\alpha = \int_a^b f\alpha' dx$$

Proof : Let $\in > 0$ be any given number.

Since f is bounded, there exists M > 0 such that

$|f(x)| \le M, \forall x \in |a,b|$

Again, since $f, a \in R[a,b]$, therefore $f a' \in R[a,b]$ and consequently there exists $\delta_1 > 0$, $\delta_2 > 0$ such that

$$\left| \sum \alpha'(t_i)\Delta x_i - \int f\alpha' dx \right| < \frac{\in}{2}$$

for $\mu(P) < \delta_2$ and all $t_i \in \Delta x - i$

Now, for $\mu(P) < \delta_2$ and all $t_i \in \Delta x_i, s_i \in x_i$, (8) gives

$$\sum |\alpha'(t_i) - \alpha'(s_i)| \Delta x_i < 2.\frac{\in}{4M} = \frac{\in}{2M}$$

Let $\delta = \min(\delta_1, \delta_2)$ and P be any partition with $\mu(P) < \delta$.

Then for all $t_i \in \Delta x_i$, by Lagrange's Mean Value Theorem, there are points $s_i \in \Delta x_i$ such that

$$\Delta\alpha_i = \alpha'(s_i)\Delta x_i$$

Thus,

$$\left| \sum f(t_i)\Delta\alpha_i - \int f\alpha' dx \right| = \left| \sum f(t_i)\alpha'(s_i)\Delta x_i - \int f\alpha' dx \right|$$

$$=$$

$$\left| \sum f(t_i)\alpha'(t_i)\Delta x_i - \int f\alpha' dx + \sum f(t_i)|\alpha'(s_i) - \alpha'(t_i)|\Delta x_i \right|$$

$$\le$$

$$\left| \sum f(t_i)\alpha'(t_i)\Delta x_i - \int f\alpha' dx \right| + \sum |f(t_i)|\,|\alpha'(s_i) - \alpha'(t_i)|\Delta x_i$$

$$< \frac{\in}{2} + M\frac{\in}{2M} = \in$$

Hence for any $\in > 0$, there exists $\delta > 0$ such that for all partitions with $\mu(P) \to \delta$, (9) holds which implies $\lim_{\mu(P)\to 0} \sum f(t_i)\,\Delta\alpha_i$ exists and equals

$$\lim_{\mu(P)\to 0} \sum f(t_i)\,\Delta\alpha_i = \int_a^b f\alpha' dx$$

i.e. $f \in R(\alpha)$ and

$$\int_a^b f d\alpha = \int_a^b f\alpha' dx$$

Theorem : If f is continuous on [a, b] and α has a continuous derivative on [a, b], then

$$\int_a^b f d\alpha = \int_a^b f\alpha' dx$$

Proof : Under the given conditions, all the integrals exist.

Let P = {a = x_0, ..., x_n = b} be any partition of [a, b]. Thus by Lagrange's Mean Value theorem it is possible to find $t_i \in (x_{i-1} - x_i)$, such that

$$\alpha(x_i) - \alpha(x_{i-1}) = \alpha'(t_i)\ (x_i - x_{i-1})\ i = 1, 2, ..., n$$

i.e. $\Delta\alpha_i = \alpha'(t_i)\Delta x_i$

Thus

$$S(P, f, \alpha) = \sum_{i-1}^{n} f(t_i)\Delta\alpha_i$$

$$= \sum_{i-1}^{n} f(t_i)\alpha'(t_i)\Delta x_i = S(P, f\alpha')$$

Proceeding to limits as $\mu(P) \to 0$ since both the limits exists, we get

$$\int_a^b f d\alpha = \int_a^b f\alpha' dx$$



12.7 LET US SUM UP

- We define Upper and Lower sum of f corresponding to partition P as follows :

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i\Delta\alpha_i$$

and

$$L(P, f, \alpha) = \sum_{i=1}^{n} M_i\Delta\alpha_i$$

- If M and m are the supremum and infimum of f in [a, b], then

$m \le m_i \le M_i \le M$

which implies

$m\Delta\alpha_i \le m_i\Delta\alpha_i \le M_i\Delta\alpha_i \le M\Delta\alpha_i$

Putting i = 1, 2, ..., n and adding all the inequalities, we get

$$m\big(\alpha(b)-\alpha(a)\big) \le L(P,f,\alpha) \le U(P,f,\alpha) \le M\big(\alpha(b)-\alpha(a)\big)$$

- For each partition P, there exist $L(P,f,\alpha)$ and $U(P,f,\alpha)$. We denote Uto be the set of all upper sums of f for all the partitions of [a, b] and L to be the set of all lower sums of f for all the partitions of [a, b]
- The infimum of the set of upper sums is called the Upper integral of f with respect to α and the supremum of the set of lower sums is called the Lower integral of f with respect to α i.e.

$$\int_a^{\bar{b}} f d\alpha = \inf U$$

and $$\int_a^b f d\alpha = \sup L$$

- The function f is said to be Integrable with respect to α in Riemann sense if

$$\int_a^{\bar{b}} f d\alpha = \int_{\underline{a}}^b f d\alpha$$

and the common value of the integral is called the Riemann-Stieltjes integral of f with respect to α, over [a, b] denoted as $\int_a^b f d\alpha$.

- The upper and lower integrals are defined for every bounded function but they may not necessarily be equal for every bounded function. Functions whose upper and lower integrals are not equal are said to be non-integrable functions.
- The Riemann-Stieltjes integral reduces to Riemann integral when α (x) = x.
- If P^* is a refinement of P, then

(i) $L(P^*,f,\alpha) \ge L(P,f,\alpha)$

(ii) $U(P^*,f,\alpha) \le U(P,f,\alpha)$

- A function f is integrable with respect to α on [a, b] if and only if for every $\in > 0$ there exists a partition P of [a, b] such that

$$U(P_2,f,\alpha) - L(P_1,f,\alpha) < \in$$

- The sum $S(P,f,\alpha)$ is defined as

$$S(P,f,\alpha) = \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i$$

corresponding to a partition P of [a, b] and $t_i \in [x_{i-1}, x_i]$

- $S(P, f, \alpha)$ coverges to A as $\mu(P) \to 0$ i.e.

  $\lim_{\mu(P)\to 0} S(P, f, \alpha) = A$

  if for every $\in > 0$, there exists $\delta > 0$ such that for every partition P of [a, b] with $\mu(P) \to \delta$ and every choice of t_i in $[x_{i-1}, x_i]$
- If $\lim_{\mu(P)\to 0} S(P, f, \alpha)$ exists, then

  $f \in R(\alpha)$ and $\lim_{\mu(P)\to 0} S(P, f, \alpha) = \int_a^b f d\alpha$
- If f is continuous on [a, b] then $f \in R(\alpha)$ over [a, b]. Also, for every $\in > 0$, there corresponds $\delta > 0$ such that

  $\left| S(P, f, \alpha) - \int_a^b f d\alpha \right| < \in$

  for every partition P = {a = x_0, ..., x_n = b}of [a, b] with $\mu(P) \to \delta$ and for every choice of $t_i \in [x_{i-1} - x_i]$ i.e.

  $\lim_{\mu(P)\to 0} S(P, f, \alpha) = \int_a^b f d\alpha$
- If $f \in R$ [a, b] and α is monotonically increasing on [a, b] such that $\alpha' \in R[a, b]$, then $f \in R(\alpha)$ and

  $\int_a^b f d\alpha = \int_a^b f\alpha' dx$
- If f is continuous on [a, b] and α has a continuous derivative on [a, b], then

  $\int_a^b f d\alpha = \int_a^b f\alpha' dx$

12.8 ANSWERS TO CHECK YOUR PROGRESS

Check your Progress 1

Ans to Q No 1: Try Yourself.

Ans to Q No 2: Try Yourself.

Check your progress 2 :

Ans to Q No 1: Let $\in > 0$ be a given positive number.

For any integer n > 0, we choose a partition P = {a = x_0, ..., x_n = b}of [a, b] such that

$$\Delta\alpha_i = \frac{\alpha(b)-\alpha(a)}{n},\ i = 1,\ 2,\ ....,\ n$$

This is possible since α is monotonically increasing as well as continuous on [a, b]. Thus α assumes every value between its bounds α (a) and α (b).

Let f be monotonically increasing on [a, b], so that its lower and upper bound m_i, M_i, in $[x_{i-1}, x_i]$ are given by

$$m_i = f(x_{i-1}),\ M_i = f(x_i),\ i = 1,2,....,n$$

Thus

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n}(M_i - m_i)\,\Delta x_i$$

$$= \frac{\alpha(b)-\alpha(a)}{n}\sum_{i=1}^{n}\{f(x_i) - f(x_{i-1})\}$$

$$= \frac{\alpha(b)-\alpha(a)}{n}\{f(b) - f(a)\}$$

$< \in$, for large n

i.e. $f \in R(\alpha)$ over [a, b]



12.9 FURTHER READING

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12.10 MODEL QUESTIONS

Q 1: A function α is such that it increase on [a, b] and is continuous at $x^{/}$ where $a \leq x^{/} \leq b$ Another function f is such that

$f(x^{/}) = 1$ and $f(x) = 0$, for $x \neq x^{/}$

Prove that $f \in R(\alpha)$ over [a, b] and $\int_a^b f d\alpha = 0$

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